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# Resonantly interacting water waves

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Coupled nonlinear equations are derived for the amplitudes of two small-amplitude resonantly interacting gravity waves in water of non-uniform depth. Such resonance is possible only for wavelengths long compared to the depth. It is shown that the same equations are obtained from the exact Euler equations, from the nonlinear shallow water theory, and from the Boussinesq equations.

## 1. Introduction

Two water waves interact resonantly if the phase function $S_2$ of one wave is exactly or nearly twice the phase function $S_1$ of the other wave. For gravity waves this is possible only if both wavelengths are long compared to the water depth. Then the two wave amplitudes satisfy a pair of coupled nonlinear equations along the common rays of the two waves, provided that the amplitudes are small. These equations have been derived by Simmons (1969) for capillary–gravity waves with constant frequencies and wavenumbers in water of infinite depth. Nayfeh (1970) derived their steady state form in water of constant finite depth, and many other authors have derived corresponding equations for three and four wave resonances. This work is reviewed in the recent book of Craik (1985).

We shall derive the amplitude equations for two resonant or nearly resonant gravity waves in water of non-uniform depth with variable frequency and variable wavenumber. First we derive them from the exact Euler equations, then again from the nonlinear shallow water theory, and finally from the Boussinesq equations. In all three cases we obtain the same equations, which shows that all three theories are applicable in the long wave regime which we consider. The derivation is based on the usual asymptotic expansion appropriate to modulated waves.

Previously, Lau & Barcilon (1972) derived amplitude equations from the Boussinesq equations. We show that their equations result as approximations to ours when the water depth is nearly constant. They were used by Bona, Boczar-Karakiewicz & Cohen (1987) in their study of sand bar formation. Bona's questions about the equations led to our work.

## 2. Formulation and derivation

Let $\phi'(x',y',z',t')$ be the velocity potential of a flow in the region bounded above by the free surface $y'=\eta'(x'z',t')$ and below by the bottom $y'=-h(\epsilon x',\epsilon z')$. We introduce the new variables

$$x=\epsilon x',\quad y=y',\quad z=\epsilon z',\quad t=\epsilon t',$$
$$\epsilon\phi(x,y,z,t)=\phi'(x',y',z',t'),\quad \epsilon\eta(x,z,t)=\eta'(x',z',t'). \tag{2.1}$$

Here $\epsilon$ is a small parameter equal to the ratio of a typical vertical scale length to a typical horizontal scale length. Then the kinematic and dynamic free surface conditions, the Laplace equation, and the kinematic bottom condition become:

$$\left.\begin{aligned} \epsilon\eta_t-\phi_y &= -\epsilon^3(\phi_x\eta_x+\phi_z\eta_z) \quad (y=\epsilon\eta), \\ \epsilon\phi_t+g\eta &= -\frac{\epsilon}{2}\phi_y^2-\frac{\epsilon^3}{2}(\phi_x^2+\phi_z^2) \quad (y=\epsilon\eta), \\ \epsilon^2(\phi_{xx}+\phi_{zz})+\phi_{yy} &= 0 \quad (-h\leqslant y\leqslant \epsilon\eta), \\ \phi_y &= -\epsilon^2(\phi_x h_x+\phi_z h_z) \quad (y=-h). \end{aligned}\right\} \tag{2.2}$$

The acceleration due to gravity is denoted $g$.

We seek a solution of (2.2) consisting of two primary waves, and their modulation products so we write

$$\eta(x,z,t)=\sum_j \mathrm{e}^{\mathrm{i}\epsilon^{-1}S_j(x,z,t)}\,[a_j(x,z,t)+\epsilon b_j(x,z,t)]+O(\epsilon^2). \tag{2.3}$$

Since $\eta$ is real the terms occur in pairs $\pm j$ with $S_j$ real, $S_{-j}=-S_j, a_{-j}=a_j^*$ and $b_{-j}=b_j^*$. Corresponding to (2.3) we seek $\phi$ in the form

$$\phi(x,y,z,t)=\sum_j \mathrm{e}^{\mathrm{i}\epsilon^{-1}S_j(x,z,t)}\left[-\frac{\mathrm{i}g}{\omega_j}a_j(x,z,t)\,\frac{\cosh k_j(y+h)}{\cosh k_j h}+\epsilon B_j(x,y,z,t)\right]+O(\epsilon^2). \tag{2.4}$$

The frequency $\omega_j(x,z,t)$ and wavenumber $k_j(x,z,t)$ of the $j$th wave are defined by

$$\omega_j=-\partial_t S_j,\quad k_j=|\nabla S_j|. \tag{2.5}$$

They are related by

$$k_j \tanh k_j h=\omega_j^2/g. \tag{2.6}$$

The form of the first term in (2.4) and (2.6) follows from the linear theory of surface waves. They could have been deduced from (2.2) but instead we shall just verify that they are correct. Reality of $\phi$ requires that $B_{-j}=B_j^*$. We assume that $a_j=0$ for $j\neq\pm 1,\pm 2$.

Equation (2.6) is a first-order partial differential equation for the phase function $S_j$. It is just the eiconal equation or dispersion equation of linear wave theory, and its characteristics are the rays.

Now we substitute (2.3) and (2.4) into (2.2) and find that the coefficients of $\epsilon^0$ vanish as a consequence of (2.6) and the form of the first term in (2.4). Then we equate to zero the coefficient of $\epsilon$ in each equation. Next we suppose that $S_2$ is nearly equal to $2S_1$, so we introduce the small 'detuning' $\delta$ defined by

$$\delta(x,z,t)=2S_1-S_2. \tag{2.7}$$

Then we equate all the terms in each equation obtained from the coefficients of $\epsilon$ which have exactly or nearly the phase $S_1$ and separately those with exactly or nearly the phase $S_2$. The solvability conditions for these two sets of equations lead to the following equations for $a_1$ and $a_2$:

$$\begin{aligned} 2\partial_t a_1-\frac{a_1}{\omega_1}\partial_t\omega_1+\left(\frac{\omega_1}{k_1^2}+\frac{gh}{\omega_1\cosh^2 k_1 h}\right)\nabla S_1\cdot\nabla a_1+\tfrac{1}{2}\omega_1\nabla\cdot\left[k_1^{-2}\left(1+\frac{2k_1 h}{\sinh 2k_1 h}\right)\nabla S_1\right]a_1 \\ =-\mathrm{i}ga_2a_1^*\,\mathrm{e}^{-\mathrm{i}\delta/\epsilon}\left\{\frac{k_2^2}{\omega_2}+\frac{k_1\cdot k_2}{\omega_1}-\frac{k_1^2}{\omega_1}-\frac{\omega_1}{g^2}(\omega_2^2-\omega_1\omega_2+\omega_1^2)\right\}. \end{aligned} \tag{2.8}$$

$$2\partial_t a_2 - \frac{a_2}{\omega_2}\partial_t \omega_2 + \left(\frac{\omega_2}{k_2^2} + \frac{gh}{\omega_2 \cosh^2 k_2 h}\right)\nabla S_2 \cdot \nabla a_2 + \tfrac{1}{2}\omega_2 \nabla \cdot \left[k_2^{-2}\left(1 + \frac{2k_2 h}{\sinh 2k_2 h}\right)\nabla S_2\right] a_2$$
$$= -\mathrm{i} g a_1^2 \,\mathrm{e}^{\mathrm{i}\delta/\epsilon}\left\{\frac{2k_1^2}{\omega_1} + \frac{\omega_2 k_1^2}{2\omega_1^2} - \frac{3\omega_1^2 \omega_2}{2g^2}\right\}. \quad (2.9)$$

## 3. Simplification of the amplitude equations

The two amplitude equations can be simplified by introducing the complex wave energy density $E(a)$ and the group velocity $C(k)$, which are defined by

$$E(a) = \tfrac{1}{2}\rho g a^2, \quad (3.1)$$

$$C(k) = \tfrac{1}{2}(gk^{-1} \tanh kh)^{\frac{1}{2}}(1 + 2kh/\sinh 2kh). \quad (3.2)$$

In (3.1) $\rho$ is the density of water. Now we multiply (2.8) by $\frac{1}{2}\rho g a_1/\omega_1$ and we can rewrite it as

$$\partial_t\left(\frac{E_1}{\omega_1}\right) + \nabla \cdot \left(\frac{E_1 C_1}{\omega_1}\frac{\nabla S_1}{k_1}\right) = -\mathrm{i} g \frac{E_1}{\omega_1} a_2 \left(\frac{a_1^*}{a_1}\right)\left\{\frac{k_2^2}{\omega_2} + \frac{k_1 \cdot k_2}{\omega_1} - \frac{k_1^2}{\omega_1} - \frac{\omega_1}{g^2}(\omega_2^2 - \omega_1\omega_2 + \omega_1^2)\right\} \mathrm{e}^{-\mathrm{i}\delta/\epsilon}. \quad (3.3)$$

Here $E_1 = E(a_1)$ and $C_1 = C(k_1)$. Similarly we can write (2.9) as

$$\partial_t\left(\frac{E_2}{\omega_2}\right) + \nabla \cdot \left(\frac{E_2 C_2}{\omega_2}\frac{\nabla S_2}{k_2}\right) = -\mathrm{i} g \frac{E_1}{\omega_2} a_2 \left\{\frac{2k_1^2}{\omega_1} + \frac{\omega_2 k_1^2}{2\omega_1^2} - \frac{3\omega_1^2\omega_2}{2g^2}\right\} \mathrm{e}^{\mathrm{i}\delta/\epsilon}. \quad (3.4)$$

When the nonlinear terms on the right-hand side are neglected, the two equations (3.3) and (3.4) are uncoupled and linear in $E_1$ and $E_2$ respectively. Then they are of the form derived by Whitham (1967), which expresses conservation of wave action. If the time derivative is omitted the linear equations are both equivalent to equation (27) of Keller (1958) for time harmonic waves.

The left-hand side of (3.3) or (3.4) can be written as follows, omitting the subscripts:

$$\partial_t\left(\frac{E}{\omega}\right) + Ck^{-1}\nabla S \cdot \nabla\left(\frac{E}{\omega}\right) + \left(\frac{E}{\omega}\right)\nabla \cdot (Ck^{-1}\nabla S) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E}{\omega}\right) + \left(\frac{E}{\omega}\right)\nabla \cdot (Ck^{-1}\nabla S). \quad (3.5)$$

Here $\mathrm{d}/\mathrm{d}t = \partial_t + Ck^{-1}\nabla S \cdot \nabla$ is just the derivative with respect to $t$ following a point moving with the group velocity $C$ in the ray direction $k^{-1}\nabla S$. Thus $\mathrm{d}/\mathrm{d}t$ is the derivative along a ray. With this interpretation, the left-hand sides of (3.3) and (3.4) become first-order ordinary differential expressions along rays. When the right-hand sides are zero, each equation is the transport equation of linear geometrical optics. With non-zero right-hand sides, they are coupled transport equations.

From (2.7) it follows that $k_2 = 2k_1 - \nabla\delta$ and $\omega_2 = 2\omega_1 - \partial_t \delta$. When $\delta = 0$, (2.6) shows that this relation is possible only if $k_1 h$ and $k_2 h$ are both small, so that tanh $k_j h \approx k_j h$. If $\nabla\delta$ and $\delta_t$ are not zero but are small it is still necessary that $k_j h$ be small. Thus resonance of a wave with its first harmonic can occur only if both waves have wavelengths which are long compared to the depth. But then the coefficients in (3.3) and (3.4) can be simplified as follows:

$$C(k) \sim (gh)^{\frac{1}{2}}, \quad (3.6)$$

$$g\left\{\frac{k_2^2}{\omega_2} + \frac{k_1 \cdot k_2}{\omega_1} - \frac{k_1^2}{\omega_1} - \frac{\omega_1}{g^2}(\omega_2^2 - \omega_1\omega_2 + \omega_1^2)\right\} \sim \frac{3\omega_1}{h}, \quad (3.7)$$

$$g\left\{\frac{2k_1^2}{\omega_1}+\frac{\omega_2 k_1^2}{2\omega_1^2}-\frac{3\omega_1^2\omega_2}{2g^2}\right\} \sim \frac{3\omega_1}{h}. \tag{3.8}$$

We now use these coefficients in (3.3) and (3.4), writing the left-hand sides in the form (3.5), to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E_1}{\omega_1}\right)+\left(\frac{E_1}{\omega_1}\right)\nabla\cdot[(gh)^{\frac{1}{2}}\hat{\boldsymbol{k}}] = -\frac{3\mathrm{i}}{h}E_1 a_2\left(\frac{a_1^*}{a_1}\right)\mathrm{e}^{-\mathrm{i}\delta/\epsilon}, \tag{3.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E_2}{\omega_2}\right)+\left(\frac{E_2}{\omega_2}\right)\nabla\cdot[(gh)^{\frac{1}{2}}\hat{\boldsymbol{k}}] = -\frac{3\mathrm{i}}{2h}E_1 a_2\,\mathrm{e}^{\mathrm{i}\delta/\epsilon}. \tag{3.10}$$

Here $\hat{\boldsymbol{k}} = \nabla S_1/|\nabla S_1|$ is a unit vector in the direction of propagation along a ray. These equations (3.9) and (3.10) are the final forms of the equations for $a_1$ and $a_2$.

When the detuning $\delta$ is constant, these equations have a solution in which $a_1$ is real and $a_2 = \mathrm{i}\tilde{a}_2\,\mathrm{e}^{\mathrm{i}\delta/\epsilon}$ where $\tilde{a}_2$ is real. For this solution we can add (3.9) to $-2\mathrm{e}^{-2\mathrm{i}\delta/\epsilon}$ times (3.10) to get the conservation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{E_1}{\omega_1}+2\frac{\tilde{E}_2}{\omega_2}\right]+\left[\frac{E_1}{\omega_1}+2\frac{\tilde{E}_2}{\omega_2}\right]\nabla\cdot[(gh)^{\frac{1}{2}}\hat{\boldsymbol{k}}] = 0, \tag{3.11}$$

where $\tilde{E}_2 = E(\tilde{a}_2)$. Since $a_1$ and $\tilde{a}_2$ are real, $E_1$ and $\tilde{E}_2$ are the actual wave energy densities of the two waves. Even when $\delta$ is not constant, (3.9) and (3.10) can be rewritten as a Hamiltonian system (Craig, personal communication).

## 4. Nonlinear shallow water theory

Equations (3.9) and (3.10) hold when the water depth is small compared to the wavelength. Therefore it is of interest to see if they can be derived from the equations of the nonlinear shallow water theory. For simplicity we shall consider only one horizontal dimension $x$. Then in this theory, the horizontal velocity $u(x,t)$ and the surface elevation $\eta(x,t)$ satisfy the equations

$$u_t+uu_x+g\eta_x = 0, \tag{4.1}$$

$$\eta_t+[u(\eta+h)]_x = 0. \tag{4.2}$$

We represent $\eta$ by the right-hand side of (2.3) multiplied by $\epsilon$, and represent $u$ similarly:

$$\eta(x,t) = \epsilon\sum_j \mathrm{e}^{\mathrm{i}\epsilon^{-1}S_j(x,t)}\,[a_j(x,t)+\epsilon b_j(x,t)]+O(\epsilon^3), \tag{4.3}$$

$$u(x,t) = \epsilon\sum_j \mathrm{e}^{\mathrm{i}\epsilon^{-1}S_j(x,t)}\,[A_j(x,t)+\epsilon B_j(x,t)]+O(\epsilon^3), \tag{4.4}$$

Now we use (4.3) and (4.4) in (4.1) and (4.2). From the leading terms, of order $\epsilon^0$, we find that $\omega_j = (gh)^{\frac{1}{2}}k_j$ and $A_j = (g/h)^{\frac{1}{2}}a_j$. The terms of order $\epsilon$ yield, after some algebra,

$$2[\partial_t a_1+(gh)^{\frac{1}{2}}\partial_x a_1]+a_1\,\partial_x(gh)^{\frac{1}{2}} = -3\mathrm{i}(g/h)^{\frac{1}{2}}k_1 a_1^* a_2\,\mathrm{e}^{-\mathrm{i}\delta/\epsilon}, \tag{4.5}$$

$$2[\partial_t a_2+(gh)^{\frac{1}{2}}\partial_x a_2]+a_2\,\partial_x(gh)^{\frac{1}{2}} = -3\mathrm{i}(g/h)^{\frac{1}{2}}k_1 a_1^2\,\mathrm{e}^{\mathrm{i}\delta/\epsilon}. \tag{4.6}$$

These are the equations for $a_1$ and $a_2$. To compare them with (3.9) and (3.10) we multiply (4.5) by $\frac{1}{2}\rho g a_1$ and (4.6) by $\frac{1}{2}\rho g a_2$ and write $\mathrm{d}/\mathrm{d}t = \partial_t+(gh)^{\frac{1}{2}}\partial_x$. Then we note that $d\omega_j/dt = 0$ as a consequence of the dispersion equation. Therefore we can write the results in the form

$$\frac{d}{dt}\frac{E_1}{\omega_1}+\frac{E_1}{\omega_1}\partial_x(gh)^{\frac{1}{2}} = -\frac{3i}{h}E_1\,a_2(a_1^*/a_1)\,e^{-i\delta/\epsilon}, \tag{4.7}$$

$$\frac{d}{dt}\frac{E_2}{\omega_2}+\frac{E_2}{\omega_2}\partial_x(gh)^{\frac{1}{2}} = -\frac{3i}{2h}E_1\,a_2\,e^{i\delta/\epsilon}. \tag{4.8}$$

These equations agree exactly with (3.9) and (3.10), specialized to the case of one horizontal dimension.

## 5. The Boussinesq equations

Lau & Barcilon (1972) considered wave interaction on the basis of a Boussinesq system of equations. If we introduce the new variables $x' = \epsilon^2 x$ and $t' = \epsilon^2 t$ into those equations and then omit the primes, the equations become

$$u_t+uu_x+g\eta_x = \epsilon^4\,[\tfrac{1}{3}h^2\,u_{xxt}+hh_x\,u_{xt}+\tfrac{1}{2}hh_{xx}\,u_t], \tag{5.1}$$

$$\eta_t+[u(\eta+h)]_x = 0. \tag{5.2}$$

Now we use (4.3) and (4.4) for $\eta$ and $u$ in (5.1) and (5.2) and proceed as before. Again we obtain (4.5) and (4.6), which lead to (4.7) and (4.8). Thus in the scaling considered here, the Boussinesq system (5.1) and (5.2) yields the same amplitude equations as do the Euler equations and the nonlinear shallow water theory.

From (5.1) and (5.2) Lau & Barcilon (1972) derived their equations (3.3) and (3.4) for two amplitudes which we shall call $A_1(x)$ and $A_2(x)$. They assumed that the $A_j$ were independent of $t$, that the $\omega_j$ were constant, and that $h = H+\epsilon Hf(x)$. With this assumption on $h$ it follows from (2.6) that $k_j = k_j^0-\frac{1}{2}\epsilon k_j^0 f(x)+O(\epsilon^2)$, where $k_j^0$ is the solution of (2.6) with $h = H$. They also wrote each wave as $A_j(x)\,e^{i\epsilon^{-1}(k_j^0 x-\omega_j t)}$ whereas we have written $a_j\,e^{i\epsilon^{-1}S_j}$. Therefore we expect that $a_j = A_j(x)\exp[i\epsilon^{-1}(k_j^0 x-\omega_j t-S_j)]$. To derive equations for $A_j$ we begin with (3.9) and (3.10), set $E_j = \frac{1}{2}\rho g a_j^2$ and use the preceding expression for $a_j$ in terms of $A_j$. When we make $x$ one-dimensional and use their assumptions on $h$, we see that the divergence terms are $O(\epsilon)$, so they can be omitted. Then we obtain

$$\frac{2(gH)^{\frac{1}{2}}}{\omega_1}[\partial_x A_1+if(x)] = \frac{-3i}{H}A_1^*\,A_2\,e^{i\Delta kx}, \tag{5.3}$$

$$\frac{2(gH)^{\frac{1}{2}}}{\omega_2}[\partial_x A_2+if(x)] = \frac{-3i}{2H}A_1^2\,e^{-i\Delta kx}. \tag{5.4}$$

Here $\Delta k = k_2^0-2k_1^0$. These equations are exactly the forms of their equations which result from simplifying their coefficients by using the fact that $\omega_j \sim k_j^0$ in their non-dimensionalization.

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